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LETTER TO THE EDITOR

Interpretation of symmetry transformations

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Abstract. A general interpretation of symmetry transformations for an arbitrary regular multidimensional differential system is given in terms of its constants of motion. The results include those of recent works in the literature as special cases and a query posed in one of them is solved.

Recently, several authors (Schwarz 1983, Sarlet 1983, Martini and Kersten 1983, Hojman 1984) have studied symmetries of mechanical systems. This work may be considered as an extension of the research of Schwarz and Martini and Kersten who treated one-dimensional linear systems only. The purpose of this letter is to provide an answer to the query posed by Schwarz (1983) at the end of his letter and to present a general approach to symmetry transformations of multidimensional regular arbitrary differential systems. The results allow a clear interpretation of the meaning of general symmetry transformations and contain, as a special case, the findings of the aforementioned authors who dealt with the one-dimensional linear case. These results further reveal that, locally and using appropriate coordinates, the algebraic structure of symmetry transformations depends only on the dimension of the problem at hand. All considerations of this letter are local in nature.

Consider the system

$$\ddot{q}^i - F^i(q^j, \dot{q}^j, t) = 0, \qquad i, j = 1, \ldots, n. \tag{1}$$

Define the infinitesimal transformation

$$q'^i = q^i + \varepsilon\eta^i(q^j, \dot{q}^j, t) + O(\varepsilon^2). \tag{2}$$

It is said that transformation (2) is a symmetry transformation of system (1) if it maps the space of solutions of (1) into itself, i.e., if η^i satisfies

$$\frac{\bar{d}}{dt}\frac{\bar{d}}{dt}\eta^i - \frac{\partial F^i}{\partial \dot{q}^j}\frac{\bar{d}}{dt}\eta^j - \frac{\partial F^i}{\partial q^j}\eta^j = 0, \tag{3}$$

to within terms of order ε^2, where

$$\bar{d}/dt \equiv F^i\partial/\partial\dot{q}^i + \dot{q}^i\partial/\partial q^i + \partial/\partial t \tag{4}$$

(see, for instance, Santilli 1978, Hojman 1984). Transformation (2) can be generalised to include a transformed time variable but there is really no advantage in doing that

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as is shown below (see also Hojman 1984). The variable η^i corresponds to W in the one-dimensional example studied by Schwarz.

It turns out that it is more convenient to study first-order systems equivalent to (1) and (2), instead of systems (1) and (2) themselves.

To this end define x^μ and $f^\mu(x^\nu)(\mu, \nu = 0, 1, \ldots, 2n)$ by

$$x^0 \equiv t, \qquad x^i \equiv q^i, \qquad x^{i+n} \equiv \dot{q}^i, \tag{5}$$

$$f^0 \equiv 1, \qquad f^i \equiv x^{i+n}, \qquad f^{i+n} \equiv F^i(x^j, x^{j+n}, t). \tag{6}$$

The system

$$\mathrm{d}x^\mu/\mathrm{d}t - f^\mu(x^\nu) = 0 \tag{7}$$

is equivalent to (1). Note that the first equation, labelled by $\mu = 0$, is an identity (see Hojman and Urrutia 1981).

The infinitesimal transformation

$$x'^\mu = x^\mu + \varepsilon\xi^\mu(x^\nu) + \mathrm{O}(\varepsilon^2) \tag{8}$$

which does include the transformation of time $t = x^0$, is said to be a symmetry transformation of system (7) if it maps the space of solutions of (7) into itself, i.e., if

$$\bar{\mathrm{d}}\xi^\mu/\mathrm{d}t - f^\mu{}_{,\nu}\xi^\nu = f^\mu\,\bar{\mathrm{d}}\xi^0/\mathrm{d}t \tag{9}$$

where

$$\bar{\mathrm{d}}/\mathrm{d}t \equiv f^\mu\partial/\partial x^\mu = f^a\partial/\partial x^a + \partial/\partial t, \qquad a = 1, 2, \ldots, 2n \tag{10}$$

(which coincides with definition (4)), and

$$A_{,\nu} \equiv \partial A/\partial x^\nu \tag{11}$$

(see Hojman and Zertuche 1984).

Note that due to the fact that time is transformed, the time derivative appearing in (7) has to be properly handled to get equations (9). It is straightforward to realise that the $\mu = 0$ component of equations (9) is an identity.

Therefore, equations (9) have only $2n$ independent components, i.e., only $2n$ of the $2n+1$ unknowns ξ^μ can be determined, which in turn means that one of them, ξ^0 say, is arbitrary.

Furthermore, due to the same fact

$$\zeta^\mu = \Lambda(x^\nu)f^\mu \tag{12}$$

solves equations (9) for arbitrary Λ.

Hence, given any solution ξ^μ to equation (9) one can always define a new solution $\bar{\xi}^\mu$

$$\bar{\xi}^\mu = \xi^\mu - \xi^0 f^\mu \tag{13}$$

taking $\Lambda = \xi^0$ (due to the linear character of (9)) such that

$$\bar{\xi}^0 = 0. \tag{14}$$

This fact justifies the possibility of considering transformations like (2) where time is *not* transformed. Equation (13) links transformations where time is transformed with those where time is left unaltered. From now on I will consider transformations of type (13) with $\xi^0 = 0$ and will drop the overbar for convenience.

It can be readily seen that equations (9) (with $\xi^0=0$) are equivalent to equations (2) with the identification

$$\xi^i \equiv \eta^i, \qquad \xi^{i+n} \equiv \bar{\mathrm{d}}\eta^i/\mathrm{d}t \tag{15}$$

(see Hojman 1984).

Consider the $2n$ functionally independent variables C^a

$$C^a = C^a(x^\nu), \qquad a = 1, \ldots, 2n, \tag{16}$$

such that

$$\det(\partial C^a/\partial x^b) \neq 0, \qquad a, b = 1, \ldots, 2n; \tag{17}$$

equation (17) means that the change of variables

$$C^\alpha = C^\alpha(x^\mu), \qquad C^0 \equiv x^0 = t, \qquad \alpha, \mu = 0, 1, \ldots, 2n, \tag{18}$$

is well defined and its inverse

$$x^\mu = x^\mu(C^\alpha) \tag{19}$$

exists, because

$$\det(\partial C^\alpha/\partial x^\mu) = \det(\partial C^a/\partial x^b) \neq 0 \tag{20}$$

(see Hojman and Urrutia 1981).

The variables C^a can always be chosen (in infinitely many ways) such that

$$\bar{\mathrm{d}}C^a/\mathrm{d}t = 0, \qquad a = 1, \ldots, 2n, \tag{21}$$

at least locally (Pontryagin 1962), i.e., so that C^a are the $2n$ functionally independent constants of motion of problem (7) (or (1)). System (7) can be written in terms of the coordinates C^α as

$$\mathrm{d}C^\alpha/\mathrm{d}t - g^\alpha(C^\beta) = 0, \qquad \alpha, \beta = 0, 1, \ldots, 2n, \tag{22}$$

where

$$g^0 = 1, \qquad g^a = 0, \qquad a = 1, \ldots, 2n, \tag{23}$$

and

$$g^\alpha = (\partial C^\alpha/\partial x^\mu) f^\mu \tag{24}$$

because of equations (21) and definition (18) (see Hojman and Urrutia 1981). Equation (22) is equivalent to equation (7) because of (20) and (24).

Consider an infinitesimal transformation

$$C'^a = C^a + \varepsilon\gamma^a(C^\beta) + \mathrm{O}(\varepsilon^i) \tag{25}$$

with C^0 kept fixed (because of the argument below equation (14)).

The equation γ^a has to satisfy, in order that (25) be a symmetry transformation for equation (22), is

$$\bar{\mathrm{d}}\gamma^a/\mathrm{d}t = 0, \tag{26}$$

which is equivalent to equation (9) taking

$$\xi^\mu = (\partial x^\mu/\partial C^\alpha)\gamma^\alpha \qquad (\gamma^0 = 0, \text{ i.e., } \xi^0 = 0), \tag{27}$$

$$f^\mu = (\partial x^\mu/\partial C^\alpha) g^\alpha. \tag{28}$$

The most general solution to equation (26) is

$$\gamma^a = \gamma^a(C^b) \tag{29}$$

where γ^a are arbitrary functions, i.e., the symmetry transformation maps constants of motion into constants of motion of the problem in question.

Therefore, the most general solution of equation (9) can be written as

$$\xi^\mu = (\partial x^\mu / \partial C^a)\gamma^a(C^b) + \xi^0(x^\nu)f^\rho) \tag{30}$$

where γ^a are arbitrary functions of the constants of motion and ξ^0 is an arbitrary function of the variables x^ν. Without losing generality one may choose $\xi^0 = 0$. In other words, the symmetry transformations may be understood as the image, in coordinate space, of the mapping of the space of constants of motion of the problem into itself. This is a unified way of understanding symmetry transformations for arbitrary multi-dimensional problems. Of course, the algebraic structure of the generators of symmetry transformations, when written in terms of the constants of motion, is universal depending only on the dimensionality of the problem, in the same sense that equations (21), (22), (23) and (26) are independent of the problem at hand. The structure of the problem is contained in the transformation laws (18), (19), (24), (27) and (28).

It is worth mentioning that the information needed for the different approaches is the same. In fact, the knowledge of two functionally independent constants of motion is necessary for the method devised by Schwarz (1983) in order to find the symmetries of the one-dimensional harmonic oscillator, while Martini and Kersten (1983) need two independent solutions of one second-order linear equation. In both cases the information is equivalent to the one needed to find the general solution of the one-dimensional problem under consideration. The information which is necessary for the method developed in this note, i.e., the knowledge of $2n$ functionally independent constants of motion of equation (1) (or (7)) also amounts to knowing the general solution of the n-dimensional problem.

Consider the one-dimensional harmonic oscillator

$$\ddot{q} + q = 0 \tag{31}$$

with general solution

$$q = C_1 \sin(t + C_2), \tag{32}$$

$$\dot{q} = C_1 \cos(t + C_2). \tag{33}$$

The constants of motion C_1 and C_2 may be written in terms of q, $\dot{q}$ and t

$$C_1 = (q^2 + \dot{q}^2)^{1/2}, \tag{34}$$

$$C_2 = \tan^{-1} q/\dot{q} - t. \tag{35}$$

The most general symmetry transformation is given by equation (30) (for $\mu = 1$, recall equation (15))

$$\eta = (\partial q / \partial C^a)\gamma^a(C^b) + \xi^0(q, \dot{q}, t)\dot{q}, \qquad a, b = 1, 2, \tag{36}$$

i.e.,

$$\eta = \sin(t + C_2)\gamma'(C^b) + C_1 \cos(t + C_2) + \xi^0 \dot{q} \tag{37}$$

or

$$\eta = q\gamma'(C^b)/C_1 + \dot{q}\gamma^2(C^b) + \xi^0 \dot{q} \tag{38}$$

which exactly coincides with the result of Schwarz (1983) (when the appropriate choice for ξ^0 is made, as in his equation (3)).

The functions γ^1 and γ^2 have an arbitrary dependence on the constants C_1 and C_2.

Similarly, it is possible to recover the results of Martini and Kersten (1983).

In summary, I have showed that for a regular multidimensional arbitrary differential system an infinite-dimensional Lie group of symmetry transformations exists, which has a universal algebraic structure (depending on its dimensionality only) in terms of the constants of motion of the problem under consideration.

The symmetry transformations may be understood as the image of coordinate space of a mapping of the space of constants of motion of the problem into itself. This interpretation provides one clear and general answer to the query posed by Schwarz (1983).

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